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Direct and Inverse Solutions of Hyperbolic Heat Conduction Problems

Ching-yu Yang*
National Kaohsiung University of Applied Sciences, Kaohsiung City 807, Taiwan, Republic of China

A sequential method for estimating boundary conditions in the field of hyperbolic heat-conduction problems is proposed. An inverse solution is deduced from a finite-difference method, the concept of future time, and a modified Newton–Raphson method. The undetermined boundary condition at each time step is denoted as an unknown variable in a set of nonlinear equations, which are formulated from the measured temperature and the calculated temperature. Then an iterative process is used to solve the set of equations. No selected function is needed to represent the undetermined function in advance. Two examples are used to demonstrate the characteristics of the proposed method. In the first example, a well-known problem is used to demonstrate the validity of the proposed direct method and then an inverse solution is evaluated. In the second example, a larger value of the relaxation time is implemented in the direct solutions and the inverse solutions. The close agreement between the exact values and the estimated results is used to confirm the validity and accuracy of the proposed method. The results show that the proposed method is an accurate and stable method for determining the boundary conditions in inverse hyperbolic heat-conduction problems.

Nomenclature

C = specific heat capacity
$\boldsymbol{J}$ = error function
k = thermal conductivity
N_t = number of the temporal measurements
p = number of grids at spatial coordinate
q = heat flux
r = number of the future time
T = temperature
T_m = value of temperature at the mth time step
t = temporal coordinate
X_m = sensitivity function of T with respect to the undetermined condition at time step m
$\boldsymbol{x}_0$ = vector of the initial guess
x = spatial coordinate
Y = measured temperature
β = relaxation time
Δ = increment of the search step
ε, δ = value of the stopping criterion
$\lambda_{i,j}$ = random number
μ = eigenvalue of matrix
ρ = density
σ = standard deviation of measurement error
$\boldsymbol{\Phi}$ = vector constructed from Φ
Φ = calculated temperature minus measured temperature
Φ_c = calculated temperature
Φ_{meas} = measured temperature
$\hat{\Phi}_u$ = component of vector $\boldsymbol{\Phi}$
ϕ_m^{q,i_q} = unknown heat flux condition at i_qth grid and mth time step
$\boldsymbol{\Psi}$ = sensitivity matrix
$\Psi_{u,v}$ = component of vector $\boldsymbol{\Psi}$

Subscripts

i, j, m, u, v = indices

Superscripts

exact = exact
meas = measured

Introduction

DURING the past few decades, many non-Fourier heat-conduction problems have been investigated. The non-Fourier effect becomes more and more attractive in practical engineering problems such as the nonhomogenous-solids-conduction process, the rapid-heating process, and the slow-conduction process.[1−14] The mathematical representation for the non-Fourier law is a hyperbolic heat-conduction equation that includes a wave-propagation term. In other words, the heat transfer propagates at a finite speed instead of the infinite speed that is parabolic heat conduction. Some researchers have studied non-Fourier heat-conduction problems. Lor and Chu[1] analyzed the problem with the interface thermal resistance. Antaki[2] discussed heat transfer in solid-phase reactions. Sanderson et al.[3] and Liu et al.[4] investigated laser-generated ultrasound models. Mullis[5,6] discussed the rapid solidification problems. Sahoo and Roetzel,[7] Roetzel and Das,[8] and Roetzel and Ranong[9] calculated the heat-exchanger problems. Lin,[10] Abdel-Hamid,[11] and Tang and Araki[12] computed the non-Fourier fin problems under periodic thermal conditions. As well, the thin film problems have also been investigated by Tan and Yang.[13,14] However, only a few works have been done on the inverse hyperbolic heat problem because the non-Fourier models induce thermal waves and lead to discontinuous temperature response.[15−18] Weber[15] used a second-order explicit difference equation to discretize the problem domain, but the relaxation time was limited to a small value ($\beta = 0.01$). Al-Khalidy[16,17] adopted a control-volume method combined with a space-marching method to solve the inverse problem ($\beta \leq 4$). Chen et al.[18] used a Laplace-transform and control-volume method combined with a nonlinear least-squares scheme to estimate the boundary.

There are two general problems in the past researches. One is that the relaxation time is confined in a limited domain and the estimation is unstable. Therefore, it is necessary to develop a robust and stable method to estimate the boundary condition in the field of the inverse hyperbolic heat-conduction problem.

In this paper, a sequential method combined with a concept of future time[19] is proposed to solve the problems step by step. As

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*Professor, Department of Mold and Die Engineering; cyyang@cc.kuas.edu.tw.

well, a modified Newton–Raphson method[20–21] is used to search for the inverse solution at each time step. In the proposed approach, the determination of the boundary conditions at each time step includes two phases: the process of direct analysis and the process of inverse analysis. In the forward-analysis process, the boundary condition is assumed as a known value and then the temperature distribution of the problem is solved. Solutions from the above process are substituted into the sensitivity analysis and integrated with the available temperature measured at the sensor's location. Thus, a set of nonlinear equations is formulated for the process of inverse estimation. In the inverse-analysis process, an iterative method is used to guide the exploring points systematically to search for the undetermined boundary condition. Then the intermediate boundary is substituted for the unknown boundary in the following analysis. Thus, several iterations are needed to obtain the undetermined boundary condition. In the present research, the proposed method formulates the problem from the difference between the calculated temperature and the one measured directly. Therefore, the inverse formulation derived from the proposed method is simpler than that derived from the nonlinear least-squares method.

This paper includes seven sections. In the first section, previous research in the field of non-Fourier conduction is introduced and the features of the proposed method are also stated. In the second section, a finite difference formulation for the hyperbolic equation is stated, and in the third section, the stability condition for the algorithm is derived. In the fourth section, the characteristics of solving the inverse problem are delineated. As well, the contents of the concept of future time, the direct problem, the sensitivity problem, and an algorithm for the proposed method are presented. Meanwhile, the criterion for stoping the iterative process is illustrated. In the fifth section, the computational algorithm for the proposed method is shown. Two examples are employed to demonstrate the proposed method in the sixth section. In the final section, the overall contribution of this research is discussed.

Problem Statement

The inverse hyperbolic heat problem consists of finding the boundary condition at one side of a medium while temperature measurements are available at the other side. Consider a slab with thickness l and constant thermal properties. This slab originally has a uniform temperature distribution. The adiabatic condition is applied to $x = 0$. At a specific time $t = 0$, a heat flux $q(t)$ is applied to $x = l$. A mathematical formation of the problem is as follows:

$$k\frac{\partial^2 T(x,t)}{\partial x^2} = \beta\rho C\frac{\partial^2 T(x,t)}{\partial t^2} + \rho C\frac{\partial T(x,t)}{\partial t} \qquad t > 0, \quad 0 < x < l \tag{1}$$

$$T(x,0) = T_0 \qquad 0 \le x \le l \tag{2}$$

$$\frac{\partial T(x,t)}{\partial t} = 0 \quad \text{at} \quad t = 0, \quad 0 \le x \le l \tag{3}$$

$$\frac{\partial T(x,t)}{\partial x} = 0 \quad \text{at} \quad x = 0, \quad t > 0 \tag{4}$$

$$q(x,t) = -k\frac{\partial T(x,t)}{\partial x} - \beta\frac{\partial q(x,t)}{\partial t} \quad \text{at} \quad x = l, \quad t > 0 \tag{5}$$

where T represents the temperature field $T(x,t)$, k is the thermal conductivity, ρC is the heat capacity per unit volume, and β is the relaxation time which is nonnegative.

The inverse problem is to estimate the boundary condition $q(l,t)$ when the temperature is measured at $x = 0$.

Direct Solution of the Hyperbolic Equations

Various analytical and numerical methods[15,22–28] have been proposed to solve hyperbolic heat-conduction problems. Weber (see Ref. 19) used finite differences to discretize the spatial and temporal domains in solving the hyperbolic equation. Carey and Tsai[22] adopted an analytic method and a finite-difference method to solve one-dimensional problems. Glass et al.[23] used MacCormack's explicit predictor–corrector scheme to evaluate a problem.[24] Tamma and Railkar[25] developed a specially tailored transfinite element to test the problems. Yang[26] used high-resolution numerical schemes to formulate the problem in an arbitrary body-fitted coordinate grid. Chen[27,28] used the Laplace transform to remove the time-derivative terms and the control-volume method to discretize the spatial coordinate for one- and two-dimensional problems.

The proposed method uses a finite difference method with equidistant grids in the spatial coordinate and the temporal coordinate. The finite difference method has been implemented in the research of Weber[15] and Carey and Tsai.[22] However, the stability condition of the described research is not clear and the relaxation time is limited to a small value. Therefore, the following derivation investigates the stable condition for solving the hyperbolic conduction equation through an eigenvalue analysis. In this study, the spatial step size is Δx and the temporal step size is Δt. The differential terms $\partial T(x,t)/\partial t$, $\partial^2 T(x,t)/\partial t^2$, and $\partial^2 T(x,t)/\partial x^2$ can be approached by Taylor series in $x = x_i$ and $t = t_j$:

$$\frac{\partial T}{\partial t}(x_i,t_j) = \frac{T(x_i,t_j+\Delta t) - T(x_i,t_j)}{\Delta t} - \frac{\Delta t}{2}\frac{\partial^2 T}{\partial t^2}(x_i,\eta_j) \tag{6}$$

$$\frac{\partial^2 T}{\partial t^2}(x_i,t_j) = \frac{T(x_i,t_j-\Delta t) - 2T(x_i,t_j) + T(x_i,t_j+\Delta t)}{\Delta t^2} - \frac{\Delta t^2}{12}\frac{\partial^4 T}{\partial t^4}(x_i,\kappa_j) \tag{7}$$

$$\frac{\partial^2 T}{\partial x^2}(x_i,t_j) = \frac{T(x_i-\Delta x,t_j) - 2T(x_i,t_j) + T(x_i+\Delta x,t_j)}{\Delta t^2} - \frac{\Delta x^2}{12}\frac{\partial^4 T}{\partial x^4}(\nu_i,\ t_j) \tag{8}$$

where $\eta_j \in (t_j, t_j+\Delta t)$, $\kappa_j \in (t_j-\Delta t, t_j+\Delta t)$, and $\nu_i \in (x_i-\Delta x, x_i+\Delta x)$.

Therefore, Eq. (1) can be discretized as follows:

$$\beta\rho C\frac{T(x_i,t_j-\Delta t) - 2T(x_i,t_j) + T(x_i,t_j+\Delta t)}{\Delta t^2} + \rho C\frac{T(x_i,t_j+\Delta t) - T(x_i,t_j)}{\Delta t} - k\frac{T(x_i-\Delta x,t_j) - 2T(x_i,t_j) + T(x_i+\Delta x,t_j)}{\Delta x^2} = \tau_{i,j} \tag{9}$$

where

$$\tau_{i,j} = \rho C\frac{\Delta t}{2}\frac{\partial^2 T}{\partial t^2}(x_i,\eta_j) + \beta\rho C\frac{\Delta t^2}{12}\frac{\partial^4 T}{\partial t^4}(x_i,\kappa_j) - k\frac{\Delta x^2}{12}\frac{\partial^4 T}{\partial x^4}(\nu_i,t_j)$$

and $\tau_{i,j}$ is the error term of the Taylor approximation.

After the error term $\tau_{i,j}$ is neglected, the difference equation is shown as

$$\beta\rho C\frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{\Delta t^2} + \rho C\frac{T_{i,j+1} - T_{i,j}}{\Delta t} - k\frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta x^2} = 0 \tag{10}$$

$$T_{i,j+1} = \lambda T_{i-1,j} + 2\left(1 - \frac{\Delta t}{2\alpha}\lambda - \lambda\right)T_{i,j} + \lambda T_{i+1,j} - \lambda\frac{\beta}{\alpha}T_{i,j-1} \tag{11}$$

where

$$\alpha = \frac{k\Delta t^2}{\rho c \Delta x^2}, \qquad \lambda = \frac{\alpha}{\beta + \Delta t} = \frac{1}{\beta + \Delta t}\frac{k\Delta t^2}{\rho c \Delta x^2}$$

$$\begin{bmatrix} T_{1,j+1} \\ T_{2,j+1} \\ \vdots \\ T_{p-1,j+1} \end{bmatrix} = \begin{bmatrix} 2\left(1 - \frac{\Delta t}{2\alpha}\lambda - \lambda\right) & \lambda & 0 & \cdots & \cdots & 0 \\ \lambda & 2\left(1 - \frac{\Delta t}{2\alpha}\lambda - \lambda\right) & \lambda & \cdots & \cdots & \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \lambda \\ 0 & \cdots & \cdots & 0 & \lambda & 2\left(1 - \frac{\Delta t}{2\alpha}\lambda - \lambda\right) \end{bmatrix}_{(p-1)\times(p-1)} \begin{bmatrix} T_{1,j} \\ T_{2,j} \\ \vdots \\ T_{p-1,j} \end{bmatrix}$$

$$-\lambda\frac{\beta}{\alpha}\begin{bmatrix} T_{1,j-1} \\ T_{2,j-1} \\ \vdots \\ T_{p-1,j-1} \end{bmatrix} + \begin{bmatrix} \lambda T_{0,j} \\ 0 \\ \vdots \\ 0 \\ \lambda T_{p,j} \end{bmatrix} \tag{12}$$

where p is the grid number of the spatial coordinate. The ith eigenvalue of the matrix is

$$\mu_i = [2 - \Delta t/(\beta + \Delta t)] - 4\lambda[\sin(i\pi/2p)]^2 \tag{13}$$

Therefore, the condition for stability is

$$\max |[2 - \Delta t/(\beta + \Delta t)] - 4\lambda[\sin(i\pi/2p)]^2| \leq 1 \quad \text{where} \quad i = 1, 2, \ldots, p-1 \tag{14}$$

$$\tfrac{1}{4}[1 - \Delta t/(\beta + \Delta t)] \leq \lambda[\sin(i\pi/2p)]^2 \leq \tfrac{1}{4}[3 - \Delta t/(\beta + \Delta t)] \tag{15}$$

The stability requires that this inequality condition hold as $\Delta x \to 0$; that is, as $p \to \infty$,

$$\lim_{p\to\infty} (\sin^2\{[(p-1)/2p]\pi\}) = 1 \tag{16}$$

Therefore, the stable interval is confined in

$$\tfrac{1}{4}[1 - \Delta t/(\beta + \Delta t)] \leq 1/(\beta + \Delta t)(k\Delta t^2/\rho c \Delta x^2) \leq \tfrac{1}{4}[3 - \Delta t/(\beta + \Delta t)] \tag{17}$$

Proposed Method to Estimate the Boundary Conditions

In each time step, an iterative algorithm is used to estimate the boundary conditions, and the temperature is measured at a different location. Some treatments are needed in the process of solving the inverse problem. There are a forward problem, a sensitivity problem, an operational algorithm, and a stopping criterion. The forward problem is used to determine the temperature distribution and the sensitivity problem is used to find the search step in the inverse problem. The operational algorithm is used to fulfill the process of inverse analysis when the solutions of the forward problem and the sensitivity problem are available. Finally, the stopping criterion is shown to stop the iterative process.

Forward Problem

The proposed method is based on a sequential algorithm and the inverse solution is solved at each time step. Therefore, Eqs. (1–5) are limited to one time step and the transient problem at $t = t_m$ is governed by the following equations:

$$k\frac{\partial^2 T(x,t_m)}{\partial x^2} = \beta\rho C\frac{\partial^2 T(x,t_m)}{\partial t^2} + \rho C\frac{\partial T(x,t_m)}{\partial t} \quad \text{at} \quad t = t_m, \quad 0 < x < l \tag{18}$$

$$T(x, t_{m-1}) = T_{m-1}, \qquad 0 \leq x \leq l \tag{19}$$

$$T(x, t_m) = T_m, \qquad 0 \leq x \leq l \tag{20}$$

$$\frac{\partial T(x,t)}{\partial x} = 0 \qquad \text{at} \quad x = 0, \qquad t = t_{m+1} \tag{21}$$

$$-k\frac{\partial T(x,t)}{\partial x} - \beta\frac{\partial q(x,t)}{\partial t} = q(x,t) = \phi^q_{m+1} \quad \text{at} \quad x = l, \qquad t = t_{m+1} \tag{22}$$

Here, the values of ϕ^q_{m+1} is denoted as the unknown boundary of the heat flux.

The inverse solution of the preceding problem is ill-posed and it is often unstable when the measured data have a slight variation in the experimental measurement. Therefore, the concept of future time is used to improve the stability of the estimation in this research. The concept of future time makes assumptions about the behavior of the experimental data at future time steps, which is included in the measurement in order to estimate the present state. In the present research, the proposed method formulates the problem from the difference between the calculated temperature and the measured one directly. As well, the equation solver is used to solve the inverse problem.

When $t = t_m$, the estimated condition between $t = t_{m-1}$ and $t = t_m$ has been evaluated and the problem is to estimate the boundary condition at $t = t_{m+1}$. To stabilize the estimated results in the inverse algorithms, in the sequential procedure it is assumed temporarilly that several future values of the estimate are constant.[19] Then the unknown conditions at the future time are equal to its present value; that is,

$$\phi^q_{m+2} = \cdots = \phi^q_{m+r-2} = \phi^q_{m+r-1} = \phi^q_{m+r} = \phi^q_{m+1} \tag{23}$$

Here r is the number of the future time.

The forward-problem equations (18–22) are solved in r steps (from $t = t_{m+1}$ to t_{m+r}) and the undetermined boundary is set by Eq. (23).

Sensitivity Problem

In the proposed method, the modified Newton–Raphson method is adopted to solve the inverse problem. Therefore, a sensitivity analysis is necessary to decide the search step. The derivative $\partial/\partial\phi_{m+1}^{q}$ is taken on both sides of Eqs. (18–22). Thus, we have

$$k\frac{\partial^2 X(x,t_m)}{\partial x^2} = \beta\rho C\frac{\partial^2 X(x,t_m)}{\partial t^2} + \rho C\frac{\partial X(x,t_m)}{\partial t} \quad \text{at} \quad t=t_m, \qquad 0<x<l \tag{24}$$

$$X(x,t_{m-1}) = 0, \qquad 0\le x\le l \tag{25}$$

$$X(x,t_m) = 0, \qquad 0\le x\le l \tag{26}$$

$$\frac{\partial X(x,t)}{\partial x} = 0 \quad \text{at} \quad x=0, \qquad t=t_{m+1} \tag{27}$$

$$-k\frac{\partial X(x,t)}{\partial x} = 1 \quad \text{at} \quad x=l, \qquad t=t_{m+1} \tag{28}$$

where

$$X_m = \frac{\partial T(x_i,t_m)}{\partial\phi_{m+1}^{q}}$$

Equations (24–28) describe the mathematical equations for the sensitivity coefficient X_m, which can be explicitly found. These equations are linear and the dependent variable X_m is related to the independent variables x and t. Therefore, the sensitive data can be determined directly through a finite difference method.

Modified Newton–Raphson Method

The Newton–Raphson method[20,21] has been widely adopted to solve sets of nonlinear equations. This method is applicable to solving nonlinear problems when the number of the equations and the number of the unknown variables are the same. In the inverse problem, the number of equations is usually larger than the number of variables; therefore a modified version of the Newton–Raphson method is necessary to deal with the inverse problem.

In the present research, the proposed method formulates the problem from the comparison between the calculated temperature and the one measured directly. Therefore, the calculated temperature $\Phi_c(\bar{i},j)$ and the measured temperature $\Phi_{\text{meas}}(\bar{i},j)$ at the i-grid of the spatial coordinate and at the j-grid of the temporal coordinate need to be evaluated first. Then the estimation of the unknown boundary at each time step can be recast as the solution of a set of nonlinear equations:

$$\Phi(\bar{i},j) = \Phi_c(\bar{i},j) - \Phi_{\text{meas}}(\bar{i},j) = 0 \tag{29}$$

where $\bar{i}=0$ and $j=m+1, m+2, \ldots, m+r$, where r is the number of future time.

The number of equations is the number of the future time r. This detailed procedure can be shown as follows:

Substitute the index j from $m+1$ to $m+r$ and the index $\bar{i}=0$; we have

$$\boldsymbol{\Phi} = [\boldsymbol{\Phi}(0,m+1), \boldsymbol{\Phi}(0,m+2), \boldsymbol{\Phi}(0,m+3), \ldots, \boldsymbol{\Phi}(0,m+r)]^T = \{\hat{\Phi}_u\} \tag{30}$$

where $\hat{\Phi}_u$ is a component of the vector $\boldsymbol{\Phi}$. The undetermined coefficients are set as follows:

$$x = \{x_v\} \tag{31}$$

where x_v is a component of the vector $\boldsymbol{x}$. The derivative of $\hat{\Phi}_u$ with respect to x_v is solved through Eqs. (24–28) and it can be expressed as follows:

$$\Psi_{u,v} = \frac{\partial\hat{\Phi}_u}{\partial x_v} \tag{32}$$

The sensitivity matrix $\boldsymbol{\Psi}$ can be defined as follows:

$$\boldsymbol{\Psi} = \{\Psi_{u,v}\} \tag{33}$$

where $u=1,2,3,\ldots,r$, $v=1$, and $\Psi_{u,v}$ is the element of $\boldsymbol{\Psi}$ at the uth row and the vth column.

With the starting vector $\boldsymbol{x}_0$ and the above derivations from Eqs. (30–33), we have the following equation:

$$\boldsymbol{x}_{\lambda+1} = \boldsymbol{x}_\lambda + \Delta_\lambda \tag{34}$$

Δ_λ is a linear least-squares solution for a set of overdetermined linear equations and it can be derived as follows:

$$\boldsymbol{\Psi}(\boldsymbol{x}_\lambda)\Delta_\lambda = -\boldsymbol{\Phi}(\boldsymbol{x}_\lambda) \tag{35}$$

$$\Delta_\lambda = -\left[\boldsymbol{\Psi}^T(\boldsymbol{x}_\lambda)\boldsymbol{\Psi}(\boldsymbol{x}_\lambda)\right]^{-1}\boldsymbol{\Psi}^T(\boldsymbol{x}_\lambda)\boldsymbol{\Phi}(\boldsymbol{x}_\lambda) \tag{36}$$

This derivation is applied at each time step. This method can be implemented in the multisensors' measurements. Under this condition, the number of elements in Eq. (30) is based on the number of measured locations and the number of future times.

Stopping Criteria

The modified Newton–Raphson method [Eqs. (34–36)] is used to determine the unknown vector $\boldsymbol{x}$ defined by Eq. (31). The step size Δ_λ goes from $\boldsymbol{x}_\lambda$ to $\boldsymbol{x}_{\lambda+l}$ and it is determined from Eq. (34). Once Δ_λ is calculated, the iterative process to determine $\boldsymbol{x}_{\lambda+l}$ is executed until the stopping criterion is satisfied.

The discrepancy principle[19] is widely used to evaluate the value of the stopping criterion in the inverse technique. However, the stopping criterion generated from the discrepancy principle does not guarantee the convergence of the inverse solution. Therefore, two criteria are chosen to assure the convergence and to stop the iteration:

$$\|\boldsymbol{x}_{\lambda+1} - \boldsymbol{x}_\lambda\| \le \delta\|\boldsymbol{x}_{\lambda+1}\| \tag{37}$$

$$\|\boldsymbol{J}(\boldsymbol{x}_{\lambda+1}) - \boldsymbol{J}(\boldsymbol{x}_\lambda)\| \le \varepsilon\|\boldsymbol{J}(\boldsymbol{x}_{\lambda+1})\| \tag{38}$$

$$\text{where} \quad \|\boldsymbol{J}(\boldsymbol{x}_{\lambda+1})\| = \sum_{i=1}^{p}\sum_{j=1}^{r}[\Phi_c(\bar{i},j) - \Phi_m(\bar{i},j)]^2 \tag{39}$$

where δ and ε are small positive values. The values of δ and ε are the convergence tolerances.

Computational Algorithm

The procedure for the proposed method can be summarized as follows: First, we choose the number of future times r, the discrete configuration of the problem domain, the temporal size Δt, the measured grid, and the estimated grid, given overall convergence tolerance δ and ε and the initial guess $\boldsymbol{x}_0$. The value of $\boldsymbol{x}_\lambda$ is known at the λth iteration. Then the iterative procedure can be summarized as follows:

Step 1: Let $j=m$ and let the temperature distributions at $\{T_{j-1}\}$ and $\{T_j\}$ be known.

Step 2: Collect the measurements $\Phi_{\text{meas}}(\bar{i},j)$, which are $Y_j^{i_1}, Y_{j+1}^{i_1}, \ldots, Y_{j+r-1}^{i_1}$.

Step 3: Assume the initial guess $\boldsymbol{x}_0$.

Step 4: Solve the forward problem [Eqs. (18–22)], and compute the calculated temperature $\Phi_c(\bar{i},j)$.

Step 5: Integrate the calculated temperature $\Phi_c(\bar{i},j)$ with the measured temperature $\Phi_{\text{meas}}(\bar{i},j)$ to construct $\boldsymbol{\Phi}$.

Step 6: Calculate the sensitivity matrix $\boldsymbol{\Psi}$ through Eqs. (24–28).

Step 7: Knowing $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$, compute the step size Δ_λ from Eq. (36).

Step 8: Knowing Δ_λ and $\boldsymbol{x}_\lambda$, compute $\boldsymbol{x}_{\lambda+l}$ from Eq. (34).

Step 9: Terminate the process if the stopping criterion [Eqs. (37) and (38)] is satisfied. Otherwise return to Step 4.

Step 10: Terminate the process if the final time step is attached. Otherwise, let $j=m+1$ return to step 2.

Results and Discussion

In this section, problems defined from Eqs. (1–5) are used as examples to estimate the unknown boundary conditions. Two examples are used to demonstrate the proposed method. In the first, a typical hyperbolic heat equation[22] with discontinuous jumped responses is illustrated. The proposed finite-difference method is used to solve the problem and it releases the constraint of Weber's approach[15] which is limited to a small value of relaxation time. The measured temperature in this example is calculated from Eqs. (1–5) when the boundary and initial conditions are known in advance. In the second, an inverse heat flux problem with the relaxation times $\beta = 0, 1, 10, 100, 1000$ is discussed. In past researches, β needed to be confined to a small value to find the inverse solution ($\beta = 0.01$ in Ref. 15:$\beta \le 4$ in Refs. 16 and 17). The estimated heat flux is imposed on one side of the slab and the temperature is measured on the other side. The simulated temperature is generated from the exact temperature in each problem and it is presumed to have measurement error. In other words, random error of measurement is added to the exact temperature. This is shown in the following equation:

$$T_{i,j}^{\text{meas}} = T_{i,j}^{\text{exact}} + \lambda_{i,j}\sigma \tag{40}$$

where the subscripts i and j are the grid numbers of the spatial and temporal coordinate, respectively. $T_{i,j}^{\text{exact}}$ in Eq. (40) is the exact temperature. $T_{i,j}^{\text{meas}}$ is the measured temperature. σ is the standard deviation of measurement error. $\lambda_{i,j}$ is a random number. The value of $\lambda_{i,j}$ is calculated by the IMSL subroutine DRNNOR[30] and chosen over the range $-2.576 < \lambda_{i,j} < 2.576$, which represents a 99% confidence bound for the measured temperature.

Example 1: Consider a finite slab subjected to the dimensionless formulation

$$\frac{\partial^2 T}{\partial t^2} + 2\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \qquad 0 \le x \le 1, \qquad t \ge 0 \tag{41}$$

$$T(x, 0) = 0 \tag{42}$$

$$\frac{\partial T}{\partial t}(x, 0) = 0 \tag{43}$$

$$T(0, t) = 1 \tag{44}$$

$$\frac{\partial T}{\partial x}(1, t) = 0 \tag{45}$$

The analytical solution of this problem has been obtained by Carey and Tsai[22] using the Laplace transform method. Numerical solutions have also been obtained by Carey and Tsai,[22] Glass et al.[23] Tamma and Railkar,[25] and Chen and Lin.[27] In this study, the sharp discontinuities of this problem can be captured by the proposed method and the direct solution is shown in Fig. 1. The oscillation appears at the discontinuity point of the temperature response when the forward-difference methods is adopted. Moreover, the central-difference and backward-difference methods have been used to solve the problem. The results[22] show that the oscillation of the numerical solution cannot be avoided in the central-difference approach and the sharp discontinuity cannot be fully expressed in the backward-difference approach. In this paper, the numerical oscillation can be reduced by a smooth process. To illustrate, the oscillation is around $x = 0.5$, which is a discontinuity point of the temperature response at $t = 0.5$ and 1.5. Furthermore, the heat wave travels from left to right at the moments $t = 0.5$ and 1. As well, the heat wave is reflected and it travels from right to left at the moment $t = 1.5$. Consequently, the numerical result confirms that the proposed finite-difference method is a valid method.

The inverse solution is shown in Fig. 2. It is assumed that the boundary is unknown [Eq. (44)] and the temperature is measured at $x = 1$. The measurement temperature errors are set between -0.05152 and 0.05152, which implies that the average standard deviation of measurements is 0.02 for a 99% confidence bound. When measurement errors are not included, the results have good approximations. When the measurement errors $\sigma = 0.02$ are included, the estimated results are still satisfied.

To investigate the deviation of the estimated results from the error-free solution, the relative average errors for the estimated solutions are defined as follows:

$$\mu = \frac{1}{N_t}\sum_{j=1}^{N_t}\left|\frac{f - \hat{f}}{\hat{f}}\right| \tag{46}$$

where f is the estimated result with measurement error and $\hat{f}$ is the estimated result without measurement error. N_t is the number of the temporal step. It is clear that a smaller value of μ indicates a better estimation and vice versa.

When measurement errors are considered, the relative average errors of the estimated results are shown in Table 1. From the results, it seems that the larger measurement error is less accurate than the smaller error. For example, the values of relative error are 0.0153813 and 0.0307625 when $\sigma = 0.01$ and $\sigma = 0.02$,

Table 1 Relative average errors of example one

Measurement error, σ	Relative error
0.01	0.0153813
0.02	0.0307625
0.03	0.0461438
0.04	0.061525
0.05	0.0769063
0.1	0.153813

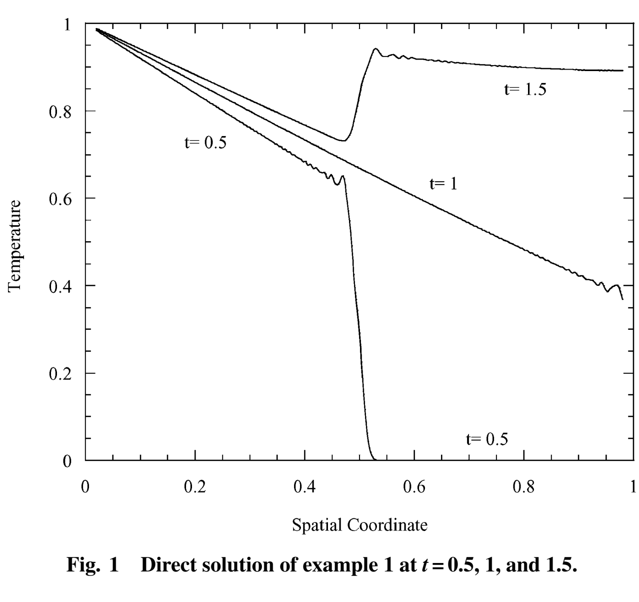

Fig. 1 Direct solution of example 1 at $t = 0.5$, 1, and 1.5.

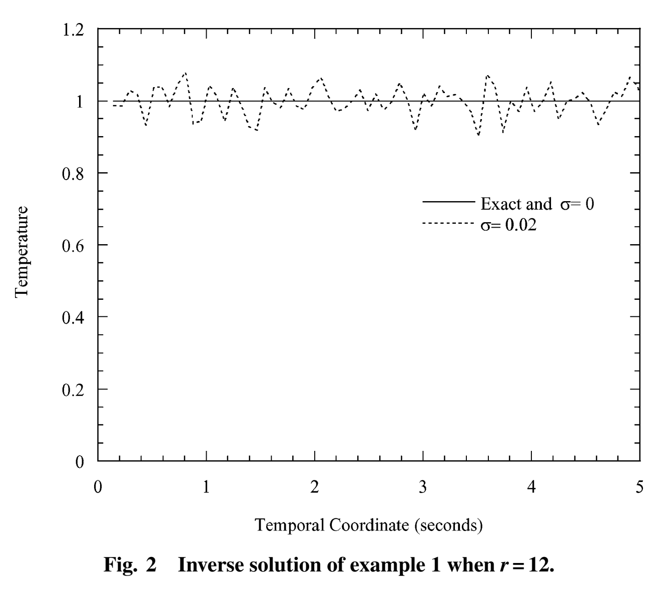

Fig. 2 Inverse solution of example 1 when $r = 12$.

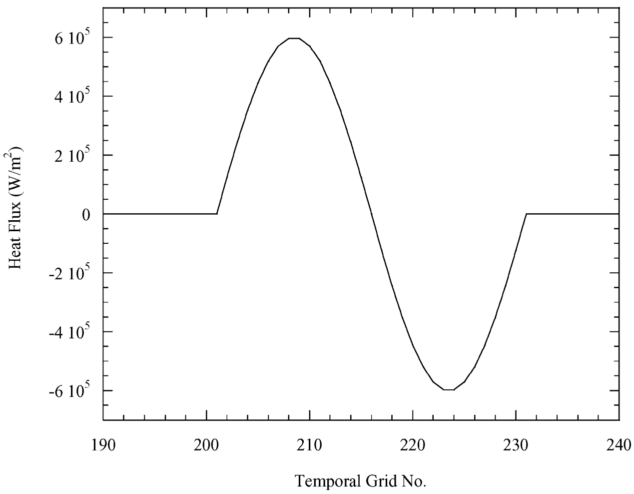

Fig. 3 Input heat flux of example 2.

respectively. Furthermore, the relative error is proportional to the measurement error with an increment of about 0.0153. This shows that the proposed method is robust and stable when the measurement error is included in the estimation.

Example 2: The inverse problem consists of finding the heat-flux input at one end while temperature is measured at the other end of the medium. Consider a slab with thickness $l = 0.035$ m and constant thermal properties $k = 50$ $\mathrm{Wm^{-2}}$ and $k/\rho c = 0.00001327\ \mathrm{m^2s^{-1}}$.

This slab originally has a uniformly distributed temperature $T_0 = 20$. A time-varying heat flux is applied at the side $x = l$ when time steps are between 201 and 229 (see Fig. 3) and the magnitude of heat flux varies with the size of Δt. The problems with different levels of $\beta(\beta = 0,\ 1,\ 10,\ 100,\ \text{and}\ 1000)$ are solved. The temperature response is calculated from time step 2 to 2000. The spatial domain is divided into 10 intervals and the size of the time step varies with the value of β defined in Eq. (17). The numerical results between temporal grids 200 and 300 are shown in Figs. 4–8. In Fig. 4, the relaxation time vanishes and the problem is of Fourier type. The Fourier heat model expresses that the heat flux is proportional

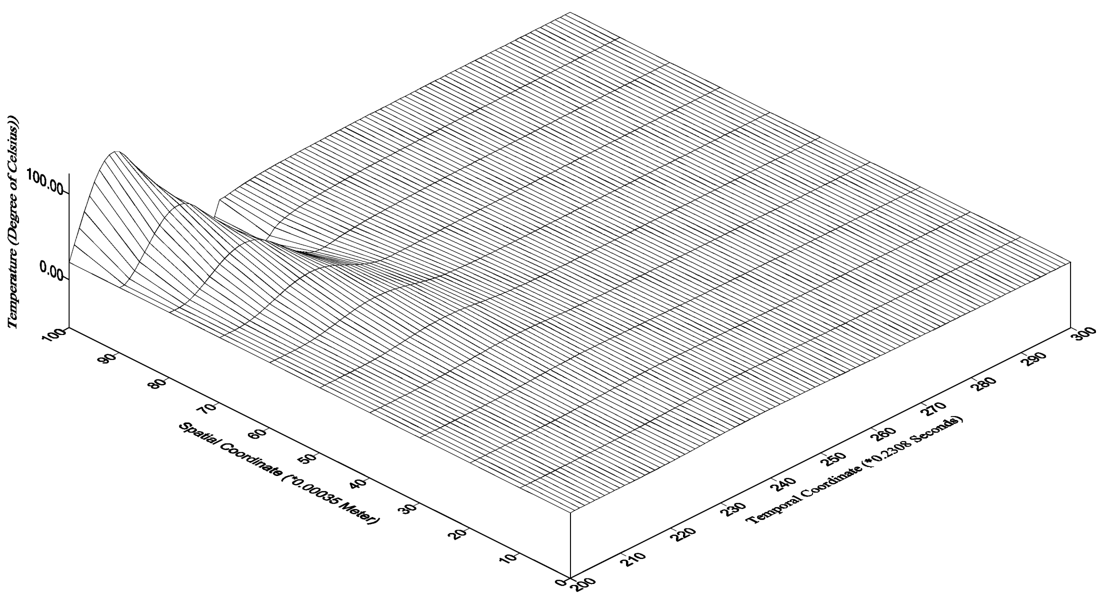

Fig. 4 Direct solution of example 2 when $\beta = 0$.

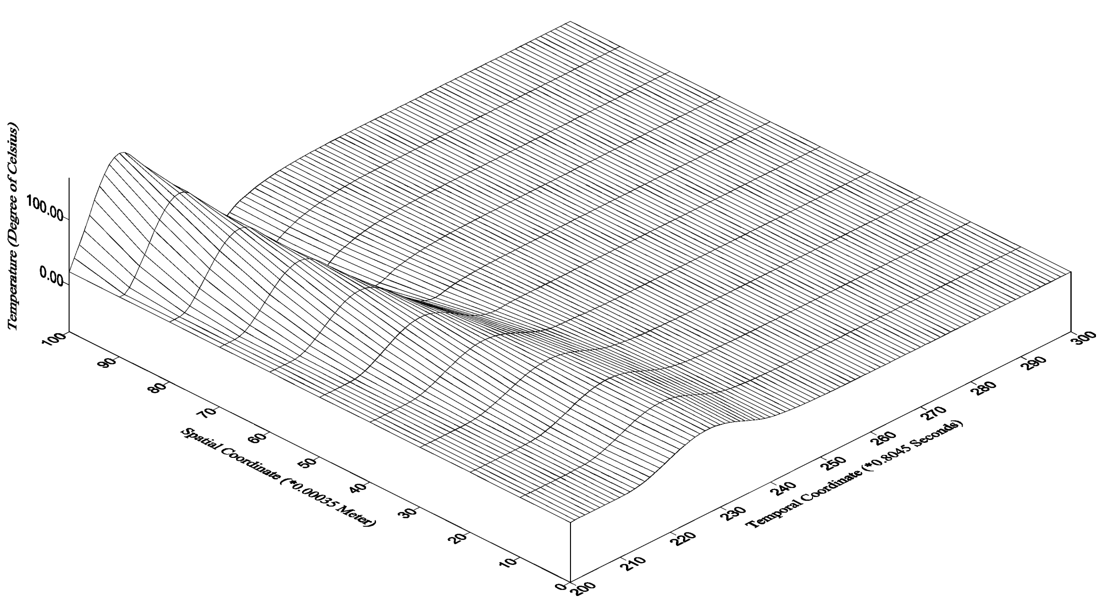

Fig. 5 Direct solution of example 2 when $\beta = 1$.

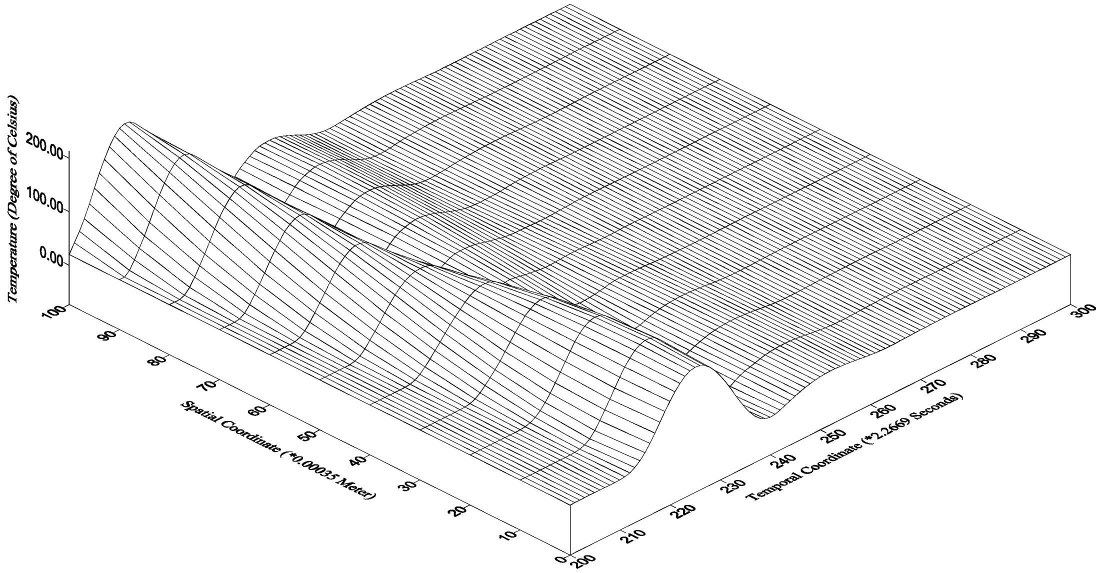

Fig. 6 Direct solution of example 2 when $\beta = 10$.

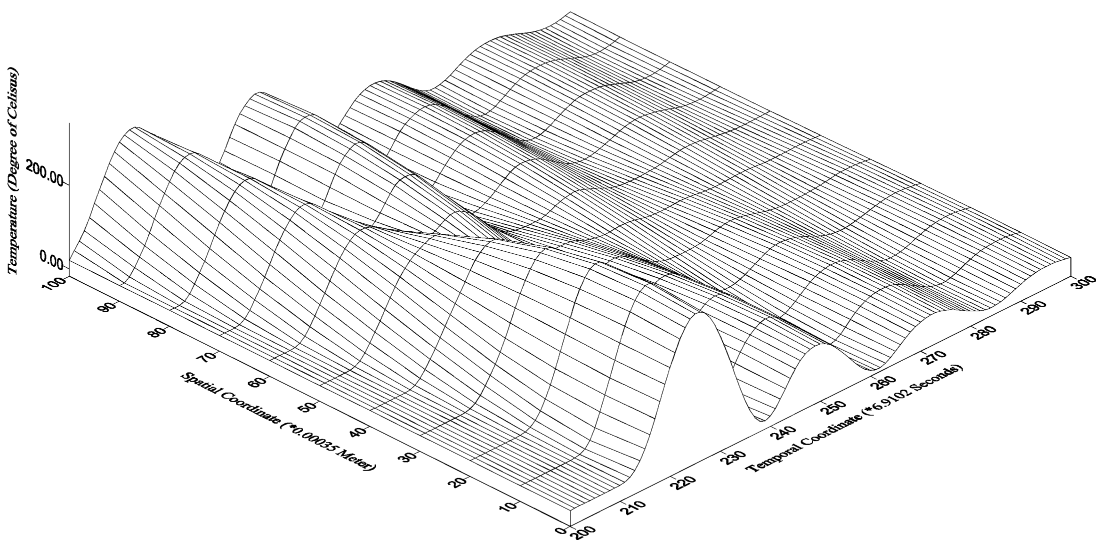

Fig. 7 Direct solution of example 2 when $\beta = 100$.

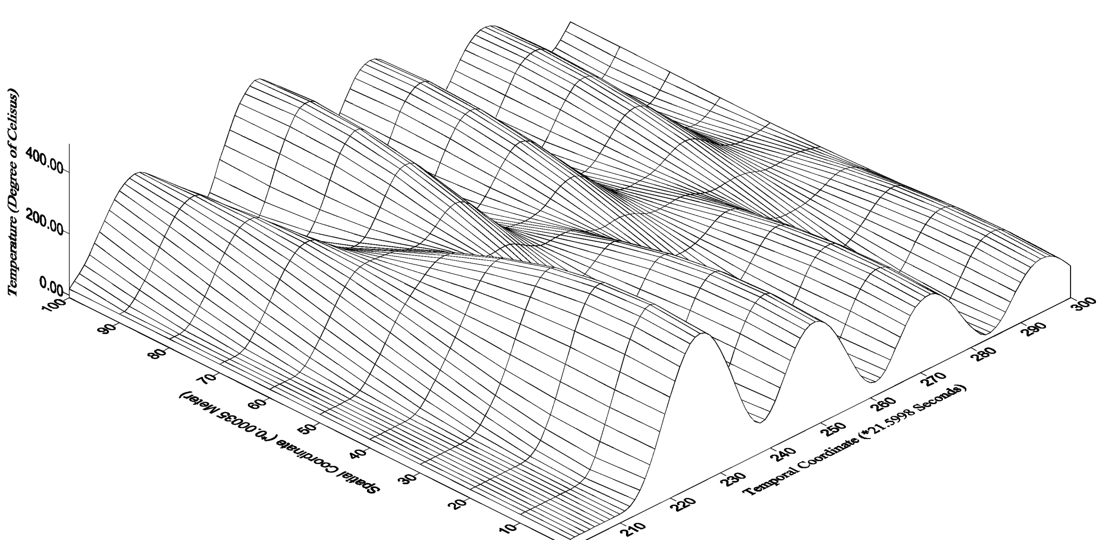

Fig. 8 Direct solution of example 2 when $\beta = 1000$.

to the temperature gradient in the model with an infinite speed of heat propagation. The results in Fig. 4 have eight time steps delay at $x = 0$, which is induced by the forward-difference method. After that, the temperature is propagated from $x = l$ to $x = 0$ immediately.

In Figs. 5–8, the relaxation time does not vanish and the problems are non-Fourier type. The non-Fourier models induce thermal waves by delaying the response between heat flux and temperature gradient. This delay may represent time needed to accumulate energy for significant heat transfer and lead to thermal wave propagation with a finite speed. In other words, the value of β postpones the heat transfer. As well, the larger values of β induce a slower thermal propagation speed. Furthermore, the frequency and amplitude of the thermal waves are also influenced by β. The results in Figs. 4 and 5 show that large values of β will induce a high frequency and amplitude based on the iterative time steps.

The inverse solution of the problems is to identify the magnitudes of the time-varying heat flux that is applied at the side $x = l$. The larger value of β needs a large size of time step to satisfy the stability condition [Eq. (17)]. Because the heat wave propagates with a finite speed, it is not expected that the input flux can be estimated from the measured temperature immediately. As well, the concept of future time is adopted to resolve the problem in order to recover the input flux from a "delay-temperature" measurement. The estimated results are shown in Figs. 9–13. An improvement of the estimation is that an accurate and stable result can be approached at the beginning of the estimation. In past researches,[16–18] the estimated results are unacceptable at the beginning of estimation. Furthermore, the application of the proposed method is not limited to a small value of β as long as the finite-difference method is stable. This is a significant improvement over the present inverse algorithms and it can extend the inverse hyperbolic problem to a more realistic saturation.

In this section, the first example demonstrates the validity of the proposed method. In the second example, the scope of the inverse hyperbolic problem can be extended to various materials with large values of β. As well, the disturbance results at the beginning of

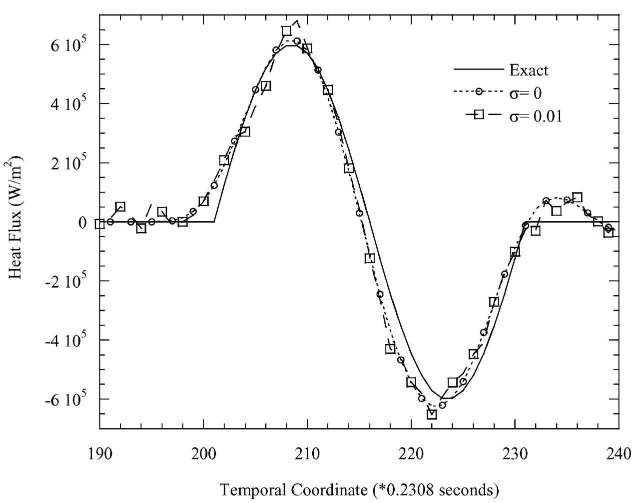

Fig. 9 Estimated results of example 2 when $\beta = 0$, $r = 13$, and $\sigma = 0$ and 0.01.

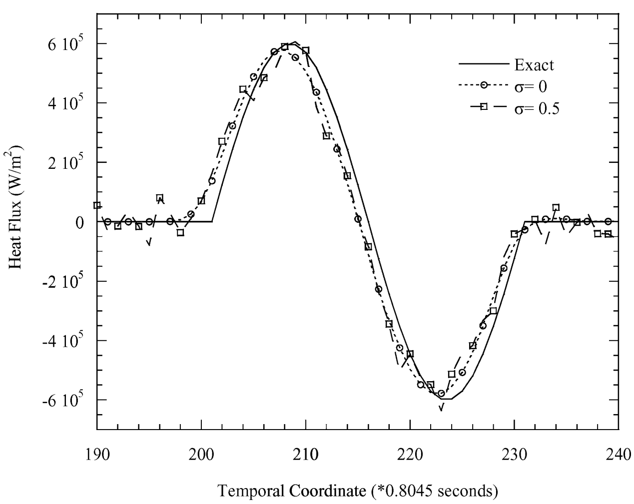

Fig. 10 Estimated results of example 2 when $\beta = 1$, $r = 12$, and $\sigma = 0$ and 0.5.

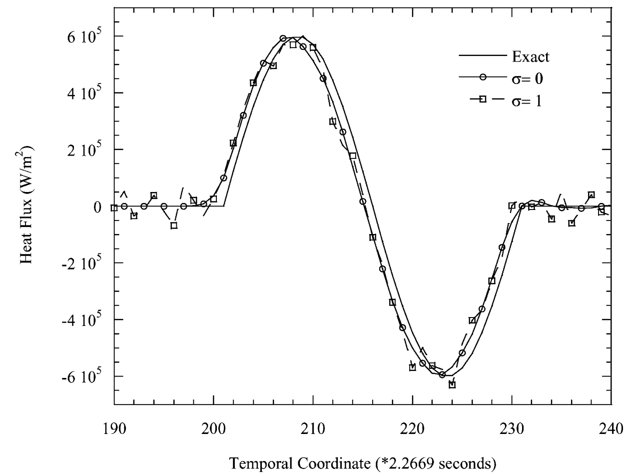

Fig. 11 Estimated results of example 2 when $\beta = 10$, $r = 11$, and $\sigma = 0$ and 1.

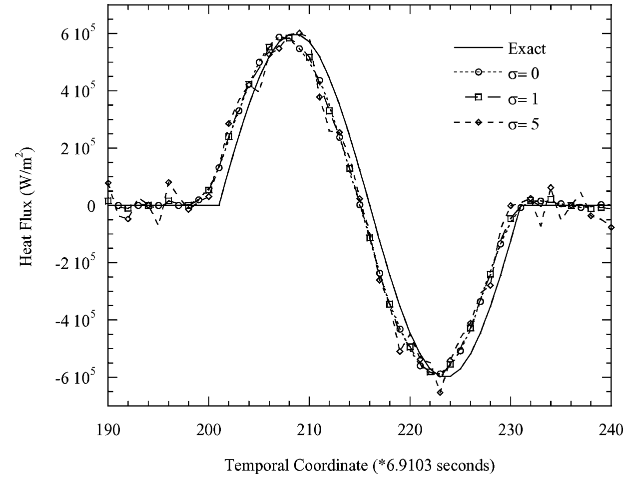

Fig. 12 Estimated results of example 2 when $\beta = 100$, $r = 12$, and $\sigma = 0$ and 5.

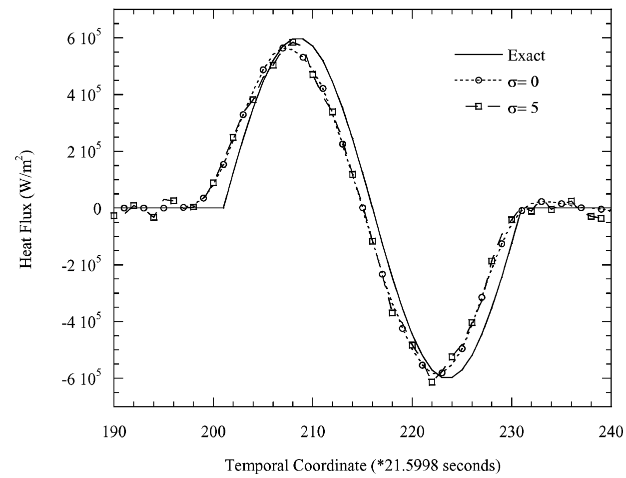

Fig. 13 Estimated results of example 2 when $\beta = 1000$, $r = 13$, and $\sigma = 0$ and 5.

the estimation can be eliminated. It is concluded that the proposed method is able to deal with the inverse hyperbolic heat-conduction problem accurately.

Conclusions

A sequential method has been introduced for determining the boundary condition in inverse hyperbolic conduction problems. The direct solution at each time step is computed by a finite difference method within a stable interval. As well, the inverse solution at each time step is solved by a modified Newton–Raphson method. The inverse method does not adopt nonlinear least-squares error to formulate the inverse problem but employs a direct comparison of the measured temperature and calculated temperature. Special features of this method are that no preselected functional form for the unknown function is necessary and no nonlinear least squares are needed in the algorithm. Two examples have been illustrated based on the proposed method. In the first example, a well-known problem with discontinuity property is solved in the direct domain and the inverse domain. In the second example, hyperbolic heat conduction with large values of relaxation time is demonstrated. The results show that the proposed method is able to find the direct and inverse solutions of the hyperbolic heat-conduction problems. In conclusion, from the results in the examples, it appears that the proposed method is an accurate and stable inverse technique. The proposed method is applicable to other kinds of inverse hyperbolic problems such as source strength estimation in the field of heat-conduction problems.

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